Macroeconomic Regimes and Regime Shifts

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Many economic time series exhibit dramatic breaks associated with events such as economic recessions, financial panics, and currency crises. Such changes in regime may arise from tipping points or other nonlinear dynamics and are core to some of the most important questions in macroeconomics. This chapter surveys the literature on regime changes. Section 1 begins with an interpretation of the move of an economy into and out of recession as an example of a change in regime and introduces some of the basic tools for analyzing such phenomena. Section 2 provides a detailed overview of econometric methods that are appropriate for time series that are subject to changes in regime. Section 3 summarizes the ways in which changes in regime can be incorporated into theoretical economic models and briefly reviews applications in a number of areas of macroeconomics.

1 Economic recessions as changes in regime.

Figure 1 plots the U.S. unemployment rate since World War II. Shaded regions highlight a feature of the data that is very familiar to macroeconomists– periodically the U.S. economy enters an episode in which the unemployment rate rises quite rapidly. These shaded regions correspond to periods that the Dating Committee of the National Bureau of Economic Research chose to designate as economic recessions. But what exactly does such a designation signify?

One view is that the statement that the economy has entered a recession does not have any intrinsic objective meaning. According to this view, the economy is always subject to unanticipated shocks, some favorable, others unfavorable. A recession is then held to be nothing more than a string of unusually bad shocks, with the bifurcation of the observed sample into periods of "recession" and "expansion" an essentially arbitrary way of summarizing the data.

Such a view is implicit in many theoretical models used in economics today insofar as it is a necessary implication of the linearity we often assume in order to make our models more tractable. But the convenience of linear models is not a good enough reason to assume that no fundamental changes in economic dynamics occur when the economy goes into a recession. For example, we understand reasonably well that in an expansion, GDP will rise more quickly at some times than others depending on the pace of new technological innovations. But what exactly would we mean by a negative technology shock? The assumption that such events are just like technological improvements but with a negative sign does not seem like the place we should start if we are trying to understand what really happens during an economic downturn.

An alternative view is that on occasion some forces that are very different from the usual technological growth take over to determine employment and output, resulting for example when a simultaneous drop in product demand across different sectors and a rapid increase in unemployed workers introduce new feedbacks of their own. The idea that there might be a tipping point at which different economic dynamics begin to take over will be a recurrent theme in this chapter.

Let's begin with a very simple model with which we can explore some of the issues. We could represent the possibility that there are two distinct phases for the economy using the random variable s_t. When $s_t = 1$, the economy is in expansion in period t and when $s_t = 2$, the economy is in recession. Suppose that an observed variable y_t such as GDP growth has an average value of $m_1 > 0$ when $s_t = 1$ and average value $m_2 < 0$ when $s_t = 2$, as in

$$y_t = m_{s_t} + \varepsilon_t \tag{1}$$

where $\varepsilon_t \sim$ i.i.d. $N(0, \sigma^2)$. Suppose that the transition between regimes is governed by a Markov chain that is independent of ε_t,

$$\text{Prob}(s_t = j | s_{t-1} = i, s_{t-2} = k, ..., y_{t-1}, y_{t-2}, ...) = p_{ij} \quad i, j = 1, 2. \tag{2}$$

Note that if both s_t and ε_t were observed directly, (1)-(2) in fact could still be described as a linear process. We can verify directly from (2) that[1]

$$E(m_{s_t} | m_{s_{t-1}}, m_{s_{t-2}}, ...) = a + \lambda m_{s_{t-1}} \tag{3}$$

where $a = p_{21}m_1 + p_{12}m_2$ and $\lambda = p_{11} - p_{21}$. In other words, m_{s_t} follows an AR(1) process,

$$m_{s_t} = a + \lambda m_{s_{t-1}} + v_t. \tag{4}$$

The innovation v_t can take on only one of 4 possible values (depending on the realization of s_t and s_{t-1}) but by virtue of (3), v_t can be characterized as a martingale difference sequence.

Suppose however that we do not observe s_t and ε_t directly, but only have observations of GDP up through date $t-1$ (denoted $\Omega_{t-1} = \{y_{t-1}, y_{t-2}, ...\}$) and want to forecast the

[1] That is,

$$E(m_{s_{t+1}} | m_{s_t} = m_1) = p_{11}m_1 + p_{12}m_2 = p_{11}m_1 + a - p_{21}m_1 = a + \lambda m_1$$

$$E(m_{s_{t+1}} | m_{s_t} = m_2) = p_{21}m_1 + p_{22}m_2 = a - p_{12}m_2 + p_{22}m_2 = a - (1 - p_{11})m_2 + (1 - p_{21})m_2 = a + \lambda m_2.$$

value of y_t. Notice from equation (1) that y_t is the sum of an AR(1) process (namely (4)) and a white noise process ε_t. Recall (e.g., Hamilton, 1994, p. 108) that the result could be described as an ARMA(1,1) process. Thus the linear projection of GDP on its own lagged values is given by

$$\hat{E}(y_t|\Omega_{t-1}) = a + \lambda y_{t-1} + \theta[y_{t-1} - \hat{E}(y_{t-1}|\Omega_{t-2})] \tag{5}$$

where θ is a known function of λ, σ^2, and the variance of v_t (Hamilton, 1994, eq. [4.7.12]). Note that we are using the notation $\hat{E}(y_t|\Omega_{t-1})$ to denote a linear projection (the forecast that produces the smallest mean squared error among the class of all linear functions of Ω_{t-1}) to distinguish it from the conditional expectation $E(y_t|\Omega_{t-1})$ (the forecast that produces the smallest mean squared error among the class of all functions of Ω_{t-1}).

Because of the discrete nature of s_t, the linear projection (5) would not yield the optimal forecast of GDP. We can demonstrate this using the law of iterated expectations (White, 1984, p. 54):

$$\begin{aligned} E(y_t|\Omega_{t-1}) &= \sum_{i=1}^{2} E(y_t|s_{t-1} = i, \Omega_{t-1})\text{Prob}(s_{t-1} = i|\Omega_{t-1}) \\ &= \sum_{i=1}^{2}(a + \lambda m_i)\text{Prob}(s_{t-1} = i|\Omega_{t-1}). \end{aligned} \tag{6}$$

Because a probability is necessarily between 0 and 1, the optimal inference $\text{Prob}(s_{t-1} = i|\Omega_{t-1})$ is necessarily a nonlinear function of Ω_{t-1}. If data through $t-1$ have persuaded us that the economy was in expansion of that point, the optimal forecast is going to be close to $a+\lambda m_1$, whereas if we become convinced the economy was in recession, the optimal forecast approaches $a+\lambda m_2$. It is in this sense that we could characterize (1) as a nonlinear process in terms of its observable implications for GDP.

Calculation of the nonlinear inference $\text{Prob}(s_{t-1} = i|\Omega_{t-1})$ is quite simple for this process. We could start for $t = 0$ for example with the ergodic probabilities of the Markov chain:

$$\text{Prob}(s_0 = 1|\Omega_0) = \frac{p_{21}}{p_{21} + p_{12}}$$

$$\text{Prob}(s_0 = 2|\Omega_0) = \frac{p_{12}}{p_{21} + p_{12}}.$$

Given a value for $\text{Prob}(s_{t-1} = i|\Omega_{t-1})$, we can arrive at the value for $\text{Prob}(s_t = j|\Omega_t)$ using Bayes's Law:

$$\text{Prob}(s_t = j|\Omega_t) = \frac{\text{Prob}(s_t = j|\Omega_{t-1})f(y_t|s_t = j, \Omega_{t-1})}{f(y_t|\Omega_{t-1})}. \tag{7}$$

Here $f(y_t|s_t = j, \Omega_{t-1})$ is the $N(m_j, \sigma^2)$ density,

$$f(y_t|s_t = j, \Omega_{t-1}) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(y_t - m_j)^2}{2\sigma^2}\right], \tag{8}$$

$\text{Prob}(s_t = j|\Omega_{t-1})$ is the predicted regime given past observations,

$$\text{Prob}(s_t = j|\Omega_{t-1}) = p_{1j}\text{Prob}(s_{t-1} = 1|\Omega_{t-1}) + p_{2j}\text{Prob}(s_{t-1} = 2|\Omega_{t-1}), \tag{9}$$

and $f(y_t|\Omega_{t-1})$ is the predictive density for GDP:

$$f(y_t|\Omega_{t-1}) = \sum_{i=1}^{2} \text{Prob}(s_t = i|\Omega_{t-1})f(y_t|s_t = i, \Omega_{t-1}). \tag{10}$$

Given a value for $\text{Prob}(s_{t-1} = i|\Omega_{t-1})$, we can thus use (7) to calculate $\text{Prob}(s_t = j|\Omega_t)$, and proceed iteratively in this fashion through the data for $t = 1, 2, ...T$ to calculate the necessary magnitude for forming the optimal nonlinear forecast given in (6).

Note that another by-product of this recursion is calculation in (10) of the predictive density for the observed data. Thus one could estimate the vector of unknown population parameters $\theta = (m_1, m_2, \sigma, p_{11}, p_{22})'$ by maximizing the log likelihood function of the observed sample of GDP growth rates,

$$\mathcal{L}(\theta) = \sum_{t=1}^{T} \log f(y_t|\Omega_{t-1}; \theta). \tag{11}$$

If the objective is to form an optimal inference about when the economy was in a recession, one can use the same principles to obtain an even better inference as more data accumulates. For example, an inference using data observed through date $t + k$ about the regime at date t is known as a k-period-ahead smoothed inference,

$$\text{Prob}(s_t = i|\Omega_{t+k}),$$

calculation of which is described in Hamilton (1994, p. 694).

Though this is a trivially simple model, it seems to do a pretty good job at capturing what is being described by the NBER's business cycle chronology. If we select only those quarters for which the NBER declared the U.S. economy to be in expansion, we calculate an average annual growth rate of 4.5%, suggesting a value for the parameter $m_1 = 4.5$. And we observe that if the NBER determined the economy to be in expansion in quarter t, 95% of the time it said the same thing in quarter $t+1$, consistent with a value of $p_{11} = 0.95$. These values implied by the NBER chronology are summarized in column 3 of Table 1. On the other hand, if we ignore the NBER dates altogether, but simply maximize the log likelihood (11) of the observed GDP data alone, we end up with very similar estimates, as seen in column 4.

Moreover, even given the challenges of data revision, the one-quarter-ahead smoothed probabilities have an excellent out-of-sample record at tracking the NBER dates. Figure 2 plots historical values for $\text{Prob}(s_t = 2|\Omega_{t+1}, \hat{\theta}_{t+1})$ where only GDP data as it was actually released as of date $t+1$ was used to estimate parameters and form the inference plotted for date t. Values before the vertical line are "simulated real-time" inferences from Chauvet and Hamilton (2006), that is, values calculated in 2005 using a separate historical real-time data vintage for each date t shown. Values after the vertical line are true real-time out-of-sample inferences as they have been published individually on www.econbrowser.com each quarter since 2005 without revision.

One attractive feature of this approach is that the linearity of the model conditional on s_t makes it almost as tractable as a fully linear model. For example, an optimal k-period-ahead forecast of GDP growth based only on observed growth through date t can be calculated immediately using (4),

$$E(y_{t+k}|\Omega_t) = \mu + \lambda^k \sum_{i=1}^{2} (m_i - \mu)\text{Prob}(s_t = i|\Omega_t) \tag{12}$$

for $\mu = a/(1-\lambda)$. Results like this make this model of changes in regime very convenient to work with.

2 Econometric treatment of changes in regime.

This section discusses a number of issues in econometric inference about data subject to changes in regime.

2.1 Multivariate or non-Gaussian processes.

Although the model in Section 1 was quite stylized, the same basic principles can be used to investigate changes in regime in much richer settings. Suppose we have a vector of variables $\mathbf{y}_t$ observed at date t and hypothesize that the density of $\mathbf{y}_t$ conditioned on its past history $\Omega_{t-1} = \{\mathbf{y}_{t-1}, \mathbf{y}_{t-2}, ...\}$ depends on parameters $\boldsymbol{\theta}$ some or all of which are different depending on the regime s_t:

$$f(\mathbf{y}_t|s_t = j, \Omega_{t-1}) = f(\mathbf{y}_t|\Omega_{t-1}; \boldsymbol{\theta}_j) \qquad \text{for } j = 1, ..., N. \tag{13}$$

In the example in Section 1, there were $N = 2$ possible regimes with $\boldsymbol{\theta}_1 = (m_1, \sigma)'$, $\boldsymbol{\theta}_2 = (m_2, \sigma)'$, and $f(\mathbf{y}_t|\Omega_{t-1}; \boldsymbol{\theta}_j)$ the $N(m_j, \sigma^2)$ density. But one could equally well consider an n-dimensional vector autoregression in which some or all of the parameters change with the regime,

$$f(\mathbf{y}_t|\Omega_{t-1}; \boldsymbol{\theta}_j) = \frac{1}{(2\pi)^{n/2}|\boldsymbol{\Sigma}_j|^{1/2}} \exp\left[-(1/2)(\mathbf{y}_t - \boldsymbol{\mu}_{jt})'\boldsymbol{\Sigma}_j^{-1}(\mathbf{y}_t - \boldsymbol{\mu}_{jt})\right]$$

$$\boldsymbol{\mu}_{jt} = \mathbf{c}_j + \boldsymbol{\Phi}_{1j}\mathbf{y}_{t-1} + \boldsymbol{\Phi}_{2j}\mathbf{y}_{t-2} + \cdots + \boldsymbol{\Phi}_{rj}\mathbf{y}_{t-r},$$

a class of models discussed in detail in Krolzig (1997). There is also no reason that a Gaussian density has to be used. For example, Dueker (1997) proposed a model of stock returns in which the innovation comes from a Student t distribution whose degrees of freedom parameter η changes with the regime.

2.2 Time-varying transition probabilities.

Nor is it necessary to assume that there are only two possible regimes or that regimes are characterized by an exogenous Markov chain. We could replace (2) with

$$\text{Prob}(s_t = j | s_{t-1} = i, s_{t-2} = k, ..., \Omega_{t-1}) = p_{ij}(\mathbf{x}_{t-1}; \boldsymbol{\lambda}) \quad i, j = 1, ..., N \tag{14}$$

where $\mathbf{x}_{t-1}$ is a subset of Ω_{t-1} or other observed variables on which one is willing to condition and $p_{ij}(\mathbf{x}_{t-1}; \boldsymbol{\lambda})$ is a specified parametric function; for examples see Diebold, Lee and Weinbach (1994), Filardo (1994), and Peria (2002). The generalization of (9) then becomes

$$\text{Prob}(s_t = j | \Omega_{t-1}) = \sum_{i=1}^{N} p_{ij}(\mathbf{x}_{t-1}; \boldsymbol{\lambda}) \text{Prob}(s_{t-1} = i | \Omega_{t-1}).$$

where the sequence $\text{Prob}(s_t = i | \Omega_t)$ can still be calculated iteratively as in (7),

$$\text{Prob}(s_t = j | \Omega_t) = \frac{\text{Prob}(s_t = j | \Omega_{t-1}) f(\mathbf{y}_t | \Omega_{t-1}; \boldsymbol{\theta}_j)}{f(\mathbf{y}_t | \Omega_{t-1})} \tag{15}$$

with the predictive density in the denominator now

$$f(\mathbf{y}_t | \Omega_{t-1}) = \sum_{i=1}^{N} \text{Prob}(s_t = i | \Omega_{t-1}) f(\mathbf{y}_t | \Omega_{t-1}; \boldsymbol{\theta}_i). \tag{16}$$

2.3 Multiple regimes.

For the special case of an exogenous Markov chain ($p_{ij}(\mathbf{x}_{t-1}; \boldsymbol{\lambda})$ does not depend on $\mathbf{x}_{t-1}$), an N-state Markov chain preserves the feature of conditional linearity. A convenient representation for exploiting this is obtained by collecting the transition probabilities in a matrix $\mathbf{P}$ whose row j, column i element corresponds to p_{ij} (so that columns of $\mathbf{P}$ sum to unity). We likewise summarize the regime at date t by an $(N \times 1)$ vector $\boldsymbol{\xi}_t$ whose ith element is unity when $s_t = i$ and zero otherwise (that is, $\boldsymbol{\xi}_t$ is given by column s_t of $\mathbf{I}_N$). Then $E(\boldsymbol{\xi}_t | s_{t-1} = i)$ has the interpretation

$$E(\boldsymbol{\xi}_t | s_{t-1} = i) = \begin{bmatrix} \text{Prob}(s_t = 1 | s_{t-1} = i) \\ \vdots \\ \text{Prob}(s_t = N | s_{t-1} = i) \end{bmatrix} = \begin{bmatrix} p_{i1} \\ \vdots \\ p_{iN} \end{bmatrix}$$

meaning

$$E(\boldsymbol{\xi}_t | \boldsymbol{\xi}_{t-1}) = \mathbf{P}\boldsymbol{\xi}_{t-1}$$

and

$$\boldsymbol{\xi}_t = \mathbf{P}\boldsymbol{\xi}_{t-1} + \mathbf{v}_t$$

for $\mathbf{v}_t$ a discrete-valued martingale difference sequence whose elements always sum to zero. Thus the Markov chain admits a VAR(1) representation, with k-period-ahead regime probabilities conditional on the observed data Ω_t given by

$$\begin{bmatrix} \text{Prob}(s_{t+k} = 1 | \Omega_t) \\ \vdots \\ \text{Prob}(s_{t+k} = N | \Omega_t) \end{bmatrix} = \mathbf{P}^k \begin{bmatrix} \text{Prob}(s_t = 1 | \Omega_t) \\ \vdots \\ \text{Prob}(s_t = N | \Omega_t) \end{bmatrix}.$$

The recursion (15) could again be started for $t = 0$ with the ergodic probabilities, which for an N-state Markov chain can be calculated as in Hamilton (1994, equation [22.2.26]). Alternative options are to take the initial probabilities as separate parameters,

$$\begin{bmatrix} \text{Prob}(s_0 = 1 | \Omega_0) \\ \vdots \\ \text{Prob}(s_0 = N | \Omega_0) \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \vdots \\ \rho_N \end{bmatrix}$$

where ρ_i could reflect prior beliefs (e.g., $\rho_1 = 1$ if the analyst knows the sample begins in regime 1), complete ignorance ($\rho_i = 1/N$ for $i = 1, ..., N$), or $\boldsymbol{\rho}$ could be a separate vector of parameters also to be chosen by maximum likelihood. Any of the last three options is particularly attractive if the EM algorithm of Hamilton (1990) is used, which often gives a very convenient and robust method for finding the value of the parameter vector $(\boldsymbol{\theta}'_1, ..., \boldsymbol{\theta}'_N, p_{ij,i=1,...,N;j=1,..,N-1}, \rho_1, ..., \rho_{N-1})'$ that maximizes the likelihood function.

Calculation of the moments and discussion of stationarity conditions for general processes subject to changes in regime can be found in Tjøstheim (1986), Yang (2000), Timmermann (2000), and Francq and Zakoïan (2001).

Although most applications assume a relatively small number of regimes, Sims and Zha (2006) used Bayesian prior information in a model with N as large as 10, while Calvet and Fisher (2004) estimated a model with thousands of regimes by imposing a functional restriction on the ways parameters vary across regimes.

2.4 Processes that depend on current and past regimes.

In the original model proposed by Hamilton (1989) for describing economic recessions, the conditional density of GDP growth y_t was presumed to depend not just on the current regime but also on the r previous regimes:

$$y_t = m_{s_t} + \phi_1(y_{t-1} - m_{s_{t-1}}) + \phi_2(y_{t-2} - m_{s_{t-2}}) + \cdots + \phi_r(y_{t-r} - m_{s_{t-r}}) + \varepsilon_t. \tag{17}$$

While at first glance this might not appear to be a special case of the general formulation given in (13), this in fact is just a matter of representing (17) using the right notation.

Taking $r = 1$ for illustration, define

$$s_t^* = \begin{cases} 1 & \text{when } s_t = 1 \text{ and } s_{t-1} = 1 \\ 2 & \text{when } s_t = 2 \text{ and } s_{t-1} = 1 \\ 3 & \text{when } s_t = 1 \text{ and } s_{t-1} = 2 \\ 4 & \text{when } s_t = 2 \text{ and } s_{t-1} = 2 \end{cases}.$$

Then s_t^* itself follows a 4-state Markov chain with transition matrix

$$\mathbf{P}^* = \begin{bmatrix} p_{11} & 0 & p_{11} & 0 \\ p_{12} & 0 & p_{12} & 0 \\ 0 & p_{21} & 0 & p_{21} \\ 0 & p_{22} & 0 & p_{22} \end{bmatrix}$$

and the model (17) can indeed be viewed as a special case of (13), with for example

$$f(y_t|s_t^* = 2, \Omega_{t-1}) = \frac{1}{(2\pi\sigma^2)^{1/2}} \exp\left\{-\frac{[y_t - m_2 - \phi_1(y_{t-1} - m_1)]^2}{2\sigma^2}\right\}.$$

2.5 Latent-variable models with changes in regime.

A more involved case that cannot be handled using the above device is if the conditional density of $\mathbf{y}_t$ depends on the full history of regimes $(s_t, s_{t-1}, ..., s_1)$ through date t. One important case in which this arises is when a process moving in and out of recession phase is proposed as a latent variable influencing an $(n \times 1)$ vector of observed variables $\mathbf{y}_t$. For example, Chauvet (1998) specified a process for an unobserved scalar business-cycle factor F_t characterized by

$$F_t = \alpha_{s_t} + \phi F_{t-1} + \eta_t$$

which influences the observed $\mathbf{y}_t$ according to

$$\mathbf{y}_t = \boldsymbol{\psi} F_t + \mathbf{q}_t$$

for $\boldsymbol{\psi}$ an $(n \times 1)$ vector of factor loadings and elements of $\mathbf{q}_t$ presumed to follow separate autoregressions. This can be viewed as a state-space model with regime-dependent parameters in which the conditional density (13) turns out to depend on the complete history $(s_t, s_{t-1}, ..., s_1)$.

One approach for handling such models is an approximation to the log likelihood and optimal inference developed by Kim (1994). Chauvet and Hamilton (2006) and Chauvet and Piger (2008) demonstrated the real-time usefulness of this approach for recognizing U.S. recessions with $\mathbf{y}_t$ a (4×1) vector of monthly indicators of sales, income, employment and industrial production, while Camacho, Perez-Quiros and Poncela (2014) have had success using a more detailed model for the Euro area.

2.6 Analysis using Bayesian methods.

Bayesian methods offer an attractive alternative to maximum likelihood estimation of parameters. The Gibbs sampler (Albert and Chib, 1993) offers a convenient approach to inference in regime-switching models that can be adapted to inference in regime-switching state-space models; see Kim and Nelson (1999) for details. Filardo and Gordon (1998) noted the usefulness of these methods for time-varying transition probabilities as well.

One caution to be aware of in applying the Gibbs sampler to regime-switching models is the role of label switching, as discussed by Celeux, Hurn and Robert (2000), Frühwirth-

Schnatter (2001), and Geweke (2007).

2.7 Selecting the number of regimes.

Often one would want to test the null hypothesis that there are N regimes against the alternative of $N+1$, and in particular to test the null hypothesis that there are no changes in regime at all ($H_0 : N = 1$). A natural idea would be to compare the values achieved for the log likelihood (11) for N and $N+1$. Unfortunately, the likelihood ratio does not have the usual asymptotic χ^2 distribution because under the null hypothesis, some of the parameters of the model become unidentified. For example, if one thought of the null hypothesis in (1) as $m_1 = m_2$, when the null is true the maximum likelihood estimates $\hat{p}_{11}$ and $\hat{p}_{22}$ do not converge to any population values. Hansen (1992), Garcia (1998), Cho and White (2007), and Carter and Steigerwald (2012) examined the distribution of the likelihood ratio statistic in this setting, though implementing their procedures can be quite involved if the model is at all complicated.

An alternative is to calculate instead general measures that trade off the fit of the likelihood against the number of parameters estimated. Popular methods such as Schwarz's (1978) Bayesian criterion rely for their asymptotic justification on the same regularity conditions whose failure causes the likelihood ratio statistic to have a nonstandard distribution. But Smith, Naik, and Tsai (2006) developed a simple test that can be used to select the number of regimes for a Markov-switching regression,

$$y_t = \mathbf{x}_t' \boldsymbol{\beta}_{s_t} + \sigma_{s_t} \varepsilon_t \tag{18}$$

where $\varepsilon_t \sim N(0,1)$ and s_t follows an N-state Markov chain. The authors proposed to estimate the parameter vector $\boldsymbol{\theta} = (\boldsymbol{\beta}'_1, ..., \boldsymbol{\beta}'_N, \sigma_1, ..., \sigma_N, p_{ij, i=1,...,N; j=1,..,N-1})'$ by maximum likelihood for each possible choice of N and calculate

$$\hat{T}_i = \sum_{t=1}^{T} \text{Prob}(s_t = i|\Omega_T; \hat{\boldsymbol{\theta}}_{MLE}) \qquad \text{for } i = 1, ..., N$$

using the full-sample smoothed probabilities. They suggested choosing the value of N for which

$$MSC = -2\mathcal{L}(\hat{\boldsymbol{\theta}}_{MLE}) + \sum_{i=1}^{N} \frac{\hat{T}_i(\hat{T}_i + Nk)}{\hat{T}_i - Nk - 2}$$

is smallest, where k is the number of elements in the regression vector $\boldsymbol{\beta}$. Other alternatives are to use Bayesian methods to find the value of N that leads to the largest value for the marginal likelihood (Chib, 1998) or the highest Bayes factor (Koop and Potter, 1999).

Another promising test of the null hypothesis of no change in regime was developed by Carrasco, Hu, and Ploberger (2014). Let $\ell_t = \log f(y_t|\Omega_{t-1}; \boldsymbol{\theta})$ be the log of the predictive density of the tth observation under the null hypothesis of no switching. For the Markov-switching regression (18), $\boldsymbol{\theta}$ would correspond to the fixed-regime regression coefficients and variance $(\boldsymbol{\beta}', \sigma^2)'$:

$$\ell_t = -(1/2)\log(2\pi\sigma^2) - \frac{(y_t - \mathbf{x}'_t\boldsymbol{\beta})^2}{2\sigma^2}.$$

Define $\mathbf{h}_t$ to be the derivative of the log density with respect to the parameter vector,

$$\mathbf{h}_t = \left.\frac{\partial \ell_t}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_0}$$

where $\hat{\boldsymbol{\theta}}_0$ denotes the MLE under the null hypothesis of no change in regime. For example,

$$\mathbf{h}_t = \begin{bmatrix} \frac{(y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}})\mathbf{x}_t}{\hat{\sigma}^2} \\ -\frac{1}{2\hat{\sigma}^2} + \frac{(y_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}})^2}{2\hat{\sigma}^4} \end{bmatrix}$$

where $\hat{\boldsymbol{\beta}} = \left(\sum_{t=1}^T \mathbf{x}_t\mathbf{x}_t'\right)^{-1}\left(\sum_{t=1}^T \mathbf{x}_t y_t\right)$ and $\hat{\sigma}^2 = T^{-1}\sum_{t=1}^T (\mathbf{y}_t - \mathbf{x}_t'\hat{\boldsymbol{\beta}})^2$. To implement the Carrasco, Hu, and Ploberger (2014) test of the null hypothesis of no change in regime against the alternative that the first element of $\boldsymbol{\beta}$ switches according to a Markov chain, let $\ell_t^{(1)}$ denote the first element of $\mathbf{h}_t$ and calculate

$$\ell_t^{(2)} = \left.\frac{\partial^2 \ell_t}{\partial \theta_1^2}\right|_{\theta = \hat{\boldsymbol{\theta}}_0}$$

$$\gamma_t(\rho) = \ell_t^{(2)} + \left[\ell_t^{(1)}\right]^2 + 2\sum_{s<t} \rho^{t-s}\ell_t^{(1)}\ell_s^{(1)}$$

where ρ is an unknown parameter characterizing the persistence of the Markov chain. We then regress $(1/2)\gamma_t(\rho)$ on $\mathbf{h}_t$, save the residuals $\hat{\varepsilon}_t(\rho)$, and calculate

$$C(\rho) = \frac{1}{2}\left[\max\left\{0, \frac{\sum_{t=1}^T \gamma_t(\rho)}{2\sqrt{\sum_{t=1}^T [\hat{\varepsilon}_t(\rho)]^2}}\right\}\right]^2.$$

We then find the value ρ^* that maximizes $C(\rho)$ over some range (e.g., $\rho \in [0.2, 0.8]$) and bootstrap to see if $C(\rho^*)$ is statistically significant. This is done by generating data with no changes in regime using the MLE $\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}_0$ and calculating $C(\rho^*)$ on each generated sample.

Another option is to conduct generic tests developed by Hamilton (1996) of the hypothesis that an N-regime model accurately describes the data. For example, if the model is correctly specified, the derivative of the log of the predictive density with respect to any element of the parameter vector,

$$\left.\frac{\partial \log p(\mathbf{y}_t|\Omega_{t-1}; \boldsymbol{\theta})}{\partial \theta_i}\right|_{\boldsymbol{\theta} = \hat{\boldsymbol{\theta}}_{MLE}},$$

should be impossible for predict from its own lagged values, a hypothesis that can be tested using simple regressions.

2.8 Deterministic breaks.

Another common approach is to treat the changes in regime as deterministic rather than random. If we wanted to test the null hypothesis of constant coefficients against the alternative that a certain subset of the coefficients of a regression switched at fixed known dates $t_1, t_2, ..., t_N$, we could do this easily enough using a standard F test (see for example Fisher, 1970). If we do not know the dates, we could calculate the value of the F statistic for every set of allowable N partitions, efficient algorithms for which have been described by Bai and Perron (2003) and Doan (2012), with critical values for interpreting the supremum of the F statistics provided by Bai and Perron (1998). Bai and Perron (1998) also described a sequential procedure with which one could first test the null hypothesis of no breaks against the alternative of $N = 1$ break, and then test $N = 1$ against $N = 2$, and so on.

Although simpler to deal with econometrically, deterministic structural breaks have the drawback that they are difficult to incorporate in a sensible way into models based on rational decision makers. Neither the assumption that people knew perfectly that the change was coming years in advance, nor the assumption that they were certain that nothing would ever change (when in the event the change did indeed appear) is very appealing. There is further the practical issue of how users of such econometric models are supposed to form their own future forecasts. Pesaran and Timmermann (2007) suggested estimating models over windows of limited subsamples, watching the data for an indication that it is time to switch to using a new model. Another drawback of interpreting structural breaks as deterministic events is that such approaches make no use of the fact that regimes such as business downturns may be a recurrent event.

2.9 Chib's multiple change-point model.

Chib (1998) offered a way to interpret multiple change-point models that gets around some of the awkward features of deterministic structural breaks. Chib's model assumes that when the process is in regime j the conditional density of the data is governed by parameter vector $\boldsymbol{\theta}_j$ as in (13). Chib assumed that the process begins at date 1 in regime $s_t = 1$ and parameter vector $\boldsymbol{\theta}_1$, and will stay there the next period with probability p_{11}. With probability $1 - p_{11}$ we get a new value $\boldsymbol{\theta}_2$, drawn perhaps from a $N(\boldsymbol{\theta}_1, \boldsymbol{\Sigma})$ distribution. Conditional on knowing that there were N such breaks, this could be viewed as a special case of an N-state Markov-switching model with transition probability matrix taking the form

$$\mathbf{P} = \begin{bmatrix} p_{11} & 0 & 0 & \cdots & 0 & 0 \\ 1-p_{11} & p_{22} & 0 & \cdots & 0 & 0 \\ 0 & 1-p_{22} & p_{33} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & p_{N-1,N-1} & 0 \\ 0 & 0 & 0 & \cdots & 1-p_{N-1,N-1} & 1 \end{bmatrix}.$$

The total number of regime changes N could then be selected using one of the methods discussed above.

Again it's not clear how to form out-of-sample forecasts with this specification. Pesaran, Pettenuzzo, and Timmermann (2006) proposed embedding Chib's model within a hierarchical prior with which one could forecast future changes in regime based on the size and duration of past breaks.

2.10 Smooth transition models.

Another econometric approach to changes in regime is the smooth transition regression model (Teräsvirta, 2004):

$$y_t = \frac{\exp[-\gamma(z_{t-1} - c)]}{1 + \exp[-\gamma(z_{t-1} - c)]}\mathbf{x}'_{t-1}\boldsymbol{\beta}_1 + \frac{1}{1 + \exp[-\gamma(z_{t-1} - c)]}\mathbf{x}'_{t-1}\boldsymbol{\beta}_2 + u_t. \tag{19}$$

Here the scalar z_{t-1} could be one of the elements of $\mathbf{x}_{t-1}$ or some known function of $\mathbf{x}_{t-1}$. For $\gamma > 0$, as $z_{t-1} \to -\infty$, the regression coefficients go to $\boldsymbol{\beta}_1$, while when $z_{t-1} \to \infty$, the regression coefficients approach $\boldsymbol{\beta}_2$. The parameter γ governs how quickly the coefficients transition as z_{t-1} crosses the threshold c.

If $\mathbf{x}_{t-1} = (y_{t-1}, y_{t-2}, ..., y_{t-r})'$, this is Teräsvirta's (1994) smooth-transition autoregression, for which typically $z_{t-1} = y_{t-d}$ for some lag d. More generally, given a data-generating process for $\mathbf{x}_t$, (19) is a fully specified time-series process for which forecasts at any horizon can be calculated by simulation. One important challenge is how to choose the lag d or more generally the switching variable z_{t-1}. Although in some settings the forecast might be similar to that coming from (6), the weights $\text{Prob}(s_{t-1} = i|\Omega_{t-1})$ in the latter would be a function of the entire history $\{y_{t-1}, y_{t-2}, ..., y_1\}$ rather than any single value.

3 Economic theory and changes in regime.

The previous section discussed econometric issues associated with analyzing series subject to changes in regime. This section reviews how these features can appear in theoretical models of the economy.

3.1 Closed-form solution of DSGE's and asset-pricing implications.

In some settings it is possible to find exact analytical solutions for a full dynamic stochastic general equilibrium model subject to changes in regime. A standard first-order condition in many macro models holds that

$$U'(C_t) = \beta E_t[U'(C_{t+1})(1 + r_{j,t+1})] \tag{20}$$

where C_t denotes consumption of a representative consumer, β a time-discount rate, and $r_{j,t+1}$ the real return on asset j between t and $t+1$. Lucas (1978) proposed a particularly simple setting in which aggregate output comes solely from nonreproducible assets (sometimes thought of as fruit coming from trees) for which equilibrium turns out to require that C_t equals the aggregate real dividend D_t paid on equities (or the annual crop of fruit). If the utility function exhibits constant relative risk aversion ($U(C) = (1+\gamma)^{-1}C^{(1+\gamma)}$, the aggregate equilibrium real stock price must satisfy

$$P_t = D_t^{-\gamma} \sum_{k=1}^{\infty} \beta^k E_t D_{t+k}^{(1+\gamma)}.$$

Since the dividend process $\{D_{t+k}\}$ is exogenous in this model, one could simply assume that the change in the log of D_t is characterized by a process such as (1). Cecchetti, Lam and

Mark (1990) used calculations related to those in (12) to find the closed-form solution for the general-equilibrium stock price,

$$P_t = \rho_{s_t} D_t,$$

where the values of ρ_1 and ρ_2 are given in equations (11) and (12) in their paper.

Lucas's assumption of an exogenous consumption and dividend process is obviously quite restrictive. Nevertheless, asset-pricing relations such as (20) have to hold regardless of how we close the rest of the model. We can always use (20) or other basic asset-pricing conditions along with an assumed process for returns to find the implications of changes in regime for financial variables in more general settings. There is a very large literature investigating these issues, covering topics such as portfolio allocation (Ang and Bekaert, 2002a; Guidolin and Timmermann, 2008), financial implications of rare-event risk (Evans, 1996; Barro, 2006), option pricing (Elliott, Chan, and Siu 2005), and the term structure of interest rates (Ang and Bekaert, 2002b; Bansal and Zhou, 2002). For a survey of this literature see Ang and Timmermann (2012).

3.2 Approximating the solution to DSGE's using perturbation methods.

First-order conditions for a much broader class of dynamic stochastic general equilibrium models with Markov regime-switching take the form

$$E_t \mathbf{a}(\mathbf{y}_{t+1}, \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}_{s_{t+1}}, \boldsymbol{\theta}_{s_t}) = \mathbf{0}. \tag{21}$$

Here $\mathbf{a}(.)$ is an $[(n_y + n_x) \times 1]$ vector-valued function, $\mathbf{y}_t$ an $(n_y \times 1)$ vector of control variables (also sometimes referred to as endogenous jump variables), $\mathbf{x}_t$ an $(n_x \times 1)$ vector of predetermined endogenous or exogenous variables, $\boldsymbol{\varepsilon}_t$ an $(n_\varepsilon \times 1)$ vector of innovations to those elements of $\mathbf{x}_t$ that are exogenous to the model, and s_t follows an N-state Markov chain. The example considered in the previous subsection is a special case of such a system with $n_y = n_x = 1$, $y_t = P_t/D_t$, $x_t = \ln(D_t/D_{t-1})$, $\theta_{s_t} = m_{s_t}$, and[2]

$$\mathbf{a}(y_{t+1}, y_t, x_t, x_{t-1}, \varepsilon_{t+1}, \varepsilon_t, m_{s_{t+1}}, m_{s_t}) = \begin{bmatrix} \beta \exp[(1+\gamma)(d_t + m_{s_{t+1}} + \varepsilon_{t+1})][(y_{t+1}+1)/y_t] - 1 \\ x_t - m_{s_t} - \varepsilon_t \end{bmatrix}.$$

For that example we were able to find closed-form solutions of the form

$$\begin{aligned} \mathbf{y}_t &= \boldsymbol{\rho}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t) \\ \mathbf{x}_t &= \mathbf{h}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t), \end{aligned}$$

namely $y_t = \rho_{s_t}$ and $x_t = m_{s_t} + \varepsilon_t$.

For more complicated models, solutions cannot be found analytically but can be approximated using the partition perturbation method developed by Foerster, et al. (2014). Their method generalizes the now-standard perturbation methods of Schmitt-Grohe and Uribe (2004) for finding linear and higher-order approximations to the solutions to DSGE's with no regime switching. Foerster, et al.'s idea is to approximate the solutions $\boldsymbol{\rho}_j(.)$ and $\mathbf{h}_j(.)$ in a neighborhood around the deterministic steady-state values satisfying $\mathbf{a}(\mathbf{y}^*, \mathbf{y}^*, \mathbf{x}^*, \mathbf{x}^*, \mathbf{0}, \mathbf{0}, \boldsymbol{\theta}^*, \boldsymbol{\theta}^*) =$

[2] Notice (20) can be written

$$D_t^\gamma = \beta E_t \left[D_{t+1}^\gamma \frac{P_{t+1} + D_{t+1}}{P_t} \right]$$

$$1 = \beta E_t \left[\left(\frac{D_{t+1}}{D_t} \right)^\gamma \left(\frac{(P_{t+1}/D_{t+1}) + 1}{P_t/D_t} \right) \left(\frac{D_{t+1}}{D_t} \right) \right].$$

$\mathbf{0}$ where $\boldsymbol{\theta}^*$ is the unconditional expectation of $\boldsymbol{\theta}_{s_t}$ calculated from the ergodic probabilities of the Markov chain,

$$\boldsymbol{\theta}^* = \sum_{j=1}^{N} \boldsymbol{\theta}_j \text{Prob}(s_t = j).$$

For the Lucas tree example from the previous subsection, $m^* = (m_1 p_{21} + m_2 p_{12})/(p_{12} + p_{21})$. We then think of a sequence of economies indexed by a continuous scalar χ such that their behavior as $\chi \to 0$ approaches the steady state, while the value at $\chi = 1$ is exactly that implied by (21):

$$\mathbf{y}_t = \boldsymbol{\rho}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi) \tag{22}$$

$$\mathbf{x}_t = \mathbf{h}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi). \tag{23}$$

As $\chi \to 0$, the randomness coming from $\boldsymbol{\varepsilon}_t$ is suppressed, and it turns out to be necessary to do the same thing for any elements of $\boldsymbol{\theta}$ that influence the steady state in order to have some fixed point around which to calculate the approximation. For elements in $\boldsymbol{\theta}_{s_t}$ that may change with regime but do not matter for the steady state, Foerster, et al. (2014) showed that it is not necessary to shrink by χ in order to approximate the dynamic solution. The authors thus specified

$$\boldsymbol{\theta}(s_t, \chi) = \begin{bmatrix} \boldsymbol{\theta}^{A*} + \chi(\boldsymbol{\theta}^A_{s_t} - \boldsymbol{\theta}^{A*}) \\ \boldsymbol{\theta}^B_{s_t} \end{bmatrix}$$

where $\boldsymbol{\theta}^A_{s_t}$ denotes the subset of elements of $\boldsymbol{\theta}_{s_t}$ that influence the steady state. The economy characterized by a particular value of χ thus needs to satisfy

$$\mathbf{0} = \int \sum_{j=1}^{N} p_{s_t,j} \mathbf{a}[\boldsymbol{\rho}_j(\mathbf{x}_t, \chi\boldsymbol{\varepsilon}_{t+1}, \chi), \mathbf{y}_t, \mathbf{x}_t, \mathbf{x}_{t-1}, \chi\boldsymbol{\varepsilon}_{t+1}, \boldsymbol{\varepsilon}_t, \boldsymbol{\theta}(j, \chi), \boldsymbol{\theta}(s_t, \chi)] dF(\boldsymbol{\varepsilon}_{t+1}) \tag{24}$$

where $F(\boldsymbol{\varepsilon}_{t+1})$ denotes the cumulative distribution function for $\boldsymbol{\varepsilon}_{t+1}$. Note (24) is satisfied by construction when evaluated at $\mathbf{y}_t = \mathbf{y}^*$, $\mathbf{x}_t = \mathbf{x}_{t-1} = \mathbf{x}^*$, $\boldsymbol{\varepsilon}_t = \mathbf{0}$, $\chi = 0$.

We next substitute (22) and (23) into (24) to arrive at a system of $N(n_y + n_x)$ equations of the form

$$\mathbf{Q}_{s_t}(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi) = \mathbf{0} \qquad s_t = 1, ..., N$$

which have to hold for all $\mathbf{x}_{t-1}$, $\boldsymbol{\varepsilon}_t$, and χ. Taking derivatives with respect to $\mathbf{x}_{t-1}$ and evaluating at $\mathbf{x}_{t-1} = \mathbf{x}^*$, $\boldsymbol{\varepsilon}_t = \mathbf{0}$, and $\chi = 0$ (that is, using a first-order Taylor approximation around the steady state) yields a system of $N(n_y + n_x)n_x$ quadratic polynomial equations in the $N(n_y + n_x)n_x$ unknowns corresponding to elements of the matrices

$$\underset{(n_y \times n_x)}{\mathbf{R}_j^x} = \left.\frac{\partial \boldsymbol{\rho}_j(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)}{\partial \mathbf{x}'_{t-1}}\right|_{\mathbf{x}_{t-1}=\mathbf{x}^*, \boldsymbol{\varepsilon}_t=\mathbf{0}, \chi=0} \qquad j = 1, ..., N$$

$$\underset{(n_x \times n_x)}{\mathbf{H}_j^x} = \left.\frac{\partial \mathbf{h}_j(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)}{\partial \mathbf{x}'_{t-1}}\right|_{\mathbf{x}_{t-1}=\mathbf{x}^*, \boldsymbol{\varepsilon}_t=\mathbf{0}, \chi=0} \qquad j = 1, ..., N.$$

The authors proposed an algorithm for finding the solution to this system of equations, that is, values for the above sets of matrices. Given these, other terms in the first-order Taylor approximation to (24) produce a system of $N(n_y + n_x)n_\varepsilon$ equations that are linear in known parameters and the unknown elements of

$$\underset{(n_y \times n_\varepsilon)}{\mathbf{R}_j^\varepsilon} = \left.\frac{\partial \boldsymbol{\rho}_j(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)}{\partial \boldsymbol{\varepsilon}'_t}\right|_{\mathbf{x}_{t-1}=\mathbf{x}^*, \boldsymbol{\varepsilon}_t=\mathbf{0}, \chi=0} \qquad j = 1, ..., N$$

$$\underset{(n_x \times n_\varepsilon)}{\mathbf{H}_j^\varepsilon} = \left.\frac{\partial \mathbf{h}_j(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)}{\partial \boldsymbol{\varepsilon}'_t}\right|_{\mathbf{x}_{t-1}=\mathbf{x}^*, \boldsymbol{\varepsilon}_t=\mathbf{0}, \chi=0} \qquad j = 1, ..., N,$$

from which $\mathbf{R}_j^\varepsilon$ and $\mathbf{H}_j^\varepsilon$ are readily calculated. Another system of $N(n_y+n_x)$ linear equations yields

$$\underset{(n_y \times 1)}{\mathbf{R}_j^{\chi}} = \left.\frac{\partial \boldsymbol{\rho}_j(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)}{\partial \chi}\right|_{\mathbf{x}_{t-1}=\mathbf{x}^*, \boldsymbol{\varepsilon}_t=\mathbf{0}, \chi=0} \qquad j = 1, ..., N$$

$$\underset{(n_x \times 1)}{\mathbf{H}_j^{\chi}} = \left.\frac{\partial \mathbf{h}_j(\mathbf{x}_{t-1}, \boldsymbol{\varepsilon}_t, \chi)}{\partial \chi}\right|_{\mathbf{x}_{t-1}=\mathbf{x}^*, \boldsymbol{\varepsilon}_t=\mathbf{0}, \chi=0} \qquad j = 1, ..., N.$$

The approximation to the solution to the regime-switching DSGE is then

$$\mathbf{y}_t = \mathbf{y}^* + \mathbf{R}_{s_t}^{x}(\mathbf{x}_{t-1} - \mathbf{x}^*) + \mathbf{R}_{s_t}^{\varepsilon}\boldsymbol{\varepsilon}_t + \mathbf{R}_{s_t}^{\chi}$$

$$\mathbf{x}_t = \mathbf{x}^* + \mathbf{H}_{s_t}^{x}(\mathbf{x}_{t-1} - \mathbf{x}^*) + \mathbf{H}_{s_t}^{\varepsilon}\boldsymbol{\varepsilon}_t + \mathbf{H}_{s_t}^{\chi}.$$

One could then go a step further if desired, taking a second-order Taylor approximation to (24). Once the first step (the linear approximation) has been completed, the second step (quadratic approximation) is actually easier to calculate numerically than the first step was, because all the second-step equations turn out to be linear in the remaining unknown magnitudes.

Lind (2014) developed an extension of this approach that could be used to form approximations to any model characterized by dramatic nonlinearities, even if regime-switching in the form of (21) is not part of the maintained structure. For example, the economic relations may change significantly when interest rates are at the zero lower bound. Lind's idea is to approximate the behavior of a nonlinear model over a set of discrete regions using relations that are linear (or possibly higher order polynomials) over individual regions, from which one can then use many of the tools discussed above for economic and econometric analysis.

3.3 Linear rational expectations models with changes in regime.

Economic researchers often use a linear special case of (21) which in the absence of regime shifts takes the form

$$\mathbf{A}E(\mathbf{y}_{t+1}|\Omega_t) = \mathbf{d} + \mathbf{B}\mathbf{y}_t + \mathbf{C}\mathbf{x}_t \tag{25}$$

$$\mathbf{x}_t = \mathbf{c} + \mathbf{\Phi}\mathbf{x}_{t-1} + \mathbf{v}_t$$

for $\mathbf{y}_t$ an $(n_y \times 1)$ vector of endogenous variables, $\Omega_t = \{\mathbf{y}_t, \mathbf{y}_{t-1}, ..., \mathbf{y}_1\}$, $\mathbf{x}_t$ an $(n_x \times 1)$ vector of exogenous variables, and $\mathbf{v}_t$ a martingale difference sequence. Such a system might have been obtained as an approximation to the first-order conditions for a nonlinear DSGE using the standard perturbation algorithm, or often is instead simply postulated as the primitive conditions of the model of interest. If $\mathbf{A}^{-1}$ exists and the number of eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ whose modulus is less than or equal to unity is equal to the number of predetermined endogenous variables, then a unique stable solution can be found of the form[3]

$$\mathbf{k}_{t+1} = \mathbf{h}_{k0} + \mathbf{H}_{kk}\mathbf{k}_t + \mathbf{H}_{kx}\mathbf{x}_t$$

$$\mathbf{d}_t = \mathbf{h}_{d0} + \mathbf{H}_{dk}\mathbf{k}_t + \mathbf{H}_{dx}\mathbf{x}_t$$

where $\mathbf{k}_t$ denotes the elements of $\mathbf{y}_t$ that correspond to predetermined variables while $\mathbf{d}_t$ collects the control or jump variables. Algorithms for finding the values of the parameters $\mathbf{h}_{i0}$ and $\mathbf{H}_{ij}$ have been developed by Blanchard and Kahn (1980), Klein (2000), and Sims (2001).

[3] Klein (2000) generalized to the case when $\mathbf{A}$ may not be invertible.

We could also generalize (25) to allow for changes in regime,

$$\mathbf{A}_{s_t} E(\mathbf{y}_{t+1}|\Omega_t, s_t, s_{t-1}, ..., s_1) = \mathbf{d}_{s_t} + \mathbf{B}_{s_t}\mathbf{y}_t + \mathbf{C}_{s_t}\mathbf{x}_t \tag{26}$$

where s_t follows an exogenous N-state Markov chain and $\mathbf{A}_j$ denotes an $(n_y \times n_y)$ matrix of parameters when the regime for date t is given by $s_t = j$. To solve such a model, Davig and Leeper (2007) suggested exploiting the feature that conditional on $\mathcal{S} = \{s_t\}_{t=1}^{\infty}$ the model is linear. Let $\mathbf{y}_{jt}$ correspond to the value of $\mathbf{y}_t$ when $s_t = j$ and collect the set of such vectors for all the possible regimes in a larger vector $\mathbf{Y}_t$:

$$\underset{(Nn_y \times 1)}{\mathbf{Y}_t} = \begin{bmatrix} \underset{(n_y \times 1)}{\mathbf{y}_{1t}} \\ \vdots \\ \underset{(n_y \times 1)}{\mathbf{y}_{Nt}} \end{bmatrix}.$$

If we restrict our consideration to solutions that satisfy the minimal-state Markov property, then

$$E(\mathbf{y}_{t+1}|\mathcal{S}, \Omega_t) = E(\mathbf{y}_{t+1}|s_{t+1}, s_t, \Omega_t)$$

and

$$E(\mathbf{y}_{t+1}|s_t = i, \Omega_t) = \sum_{j=1}^{N} E(\mathbf{y}_{t+1}|s_{t+1} = j, s_t = i, \Omega_t)p_{ij}.$$

Hence when $s_t = i$,

$$\mathbf{A}_{s_t} E(\mathbf{y}_{t+1}|s_t, \Omega_t) = (\mathbf{p}_i' \otimes \mathbf{A}_i)E(\mathbf{Y}_{t+1}|\mathbf{Y}_t) \tag{27}$$

where

$$\mathbf{p}_i = \begin{bmatrix} p_{i1} \\ \vdots \\ p_{iN} \end{bmatrix}$$

denotes column i of the Markov transition probabilities, with elements of $\mathbf{p}_i$ summing to unity. Consider then the stacked structural system,

$$\mathbf{A}E(\mathbf{Y}_{t+1}|\mathbf{Y}_t) = \mathbf{d} + \mathbf{B}\mathbf{Y}_t + \mathbf{C}\mathbf{x}_t \tag{28}$$

$$\underset{(Nn_y \times Nn_y)}{\mathbf{A}} = \begin{bmatrix} \underset{(1\times N)}{\mathbf{p}_1^{'}} \otimes \underset{(n_y\times n_y)}{\mathbf{A}_1} \\ \vdots \\ \underset{(1\times N)}{\mathbf{p}_N^{'}} \otimes \underset{(n_y\times n_y)}{\mathbf{A}_N} \end{bmatrix} \qquad \underset{(Nn_y \times 1)}{\mathbf{d}} = \begin{bmatrix} \underset{(n_y\times 1)}{\mathbf{d}_1} \\ \vdots \\ \underset{(n_y\times 1)}{\mathbf{d}_N} \end{bmatrix} \tag{29}$$

$$\underset{(Nn_y \times Nn_y)}{\mathbf{B}} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{B}_2 & \cdots & \mathbf{0} \\ \vdots & \vdots & \cdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_N \end{bmatrix} \qquad \underset{(Nn_y \times n_x)}{\mathbf{C}} = \begin{bmatrix} \underset{(n_y\times n_x)}{\mathbf{C}_1} \\ \vdots \\ \underset{(n_y\times n_x)}{\mathbf{C}_N} \end{bmatrix}.$$

This is a simple regime-independent system for which a solution can be found using the traditional method. For example, with no predetermined variables, if all eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ are outside the unit circle, then we can find a unique stable solution of the form

$$\underset{(Nn_y\times 1)}{\mathbf{Y}_t} = \underset{(Nn_y\times 1)}{\mathbf{h}} + \underset{(Nn_y\times n_x)}{\mathbf{H}}\underset{(n_x\times 1)}{\mathbf{x}_t} \tag{30}$$

which implies that

$$\underset{(n_y\times 1)}{\mathbf{y}_t} = \underset{(n_y\times 1)}{\mathbf{h}_{s_t}} + \underset{(n_y\times n_x)}{\mathbf{H}_{s_t}}\underset{(n_x\times 1)}{\mathbf{x}_t} \tag{31}$$

for $\mathbf{h}_i$ and $\mathbf{H}_i$ the ith blocks of $\mathbf{h}$ and $\mathbf{H}$, respectively. If (30) is a solution to (28), then (31) is a solution to (26).[4]

[4] If (31) holds, then

$$E(\mathbf{y}_{t+1}|\Omega_t, s_t = i) = \sum_{j=1}^{N} p_{ij}[\mathbf{h}_j + \mathbf{H}_j(\mathbf{c} + \mathbf{\Phi}\mathbf{x}_t)].$$

However, Farmer, Waggoner, and Zha (2010) demonstrated that while (30) yields one stable solution to (26), it need not be the only stable solution. For further discussion see Farmer, Waggoner and Zha (2009).

3.4 Multiple equilibria.

Other economists have argued that models in which there are multiple possible solutions– for example, system (25) with no predetermined variables and an eigenvalue of $\mathbf{A}^{-1}\mathbf{B}$ inside the unit circle– are precisely those we should be most interested in, given the perception that sometimes consumers or firms seem to become highly pessimistic for no discernible reason, bringing the economy into a self-fulfilling downturn; see Benhabib and Farmer (1999) for a survey of this literature. One factor that could produce multiple equilibria is coordination externalities. The rewards to me of participating in a market may be greatest when I expect large numbers of others to do the same (Cooper and John, 1988; Cooper, 1994). Multiple equilibria could also arise when expectations themselves are a factor determining

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Thus for (26) to hold it must be the case that for each $i = 1, ..., N$,

$$\mathbf{A}_i \sum_{j=1}^{N} p_{ij}\mathbf{H}_j\mathbf{\Phi} = \mathbf{B}_i\mathbf{H}_i + \mathbf{C}_i$$

$$\mathbf{A}_i \sum_{j=1}^{N} p_{ij}(\mathbf{h}_j + \mathbf{H}_j\mathbf{c}) = \mathbf{d}_i + \mathbf{B}_i\mathbf{h}_i.$$

But if (30) is a solution to (28), then

$$\mathbf{A}[\mathbf{h} + \mathbf{H}(\mathbf{c} + \mathbf{\Phi}\mathbf{x}_t)] = \mathbf{d} + \mathbf{B}(\mathbf{h} + \mathbf{H}\mathbf{x}_t) + \mathbf{C}\mathbf{x}_t$$

block i of which requires from (29) that

$$(\mathbf{p}_i' \otimes \mathbf{A}_i)\mathbf{H}\mathbf{\Phi} = \mathbf{B}_i\mathbf{H}_i + \mathbf{C}_i$$

$$(\mathbf{p}_i' \otimes \mathbf{A}_i)(\mathbf{h} + \mathbf{H}\mathbf{c}) = \mathbf{d}_i + \mathbf{B}_i\mathbf{h}_i$$

as were claimed to hold.

the equilibrium (Kurz and Motolese, 2001). Kirman (1993) and Chamley (1999) discussed mechanisms by which the economy might tend to oscillate periodically between the possible regimes in multiple-equilibria settings.

A widely studied example is financial market bubbles. In the special case of risk-neutral investors (that is, when $U'(C)$ is some constant independent of consumption C), equation (20) relating the price of the stock P_t to its future dividend D_{t+1} becomes

$$P_t = \beta E_t(P_{t+1} + D_{t+1}). \tag{32}$$

One solution to (32) is the market-fundamentals solution given by

$$P_t^* = \sum_{j=1}^{\infty} \beta^j E_t(D_{t+j}).$$

But $P_t = P_t^* + B_t$ also satisfies (32) for B_t any bubble process satisfying $B_t = \beta E_t B_{t+1}$. Hall, Psaradakis and Sola (1999) proposed an empirical test of whether an observed financial price is occasionally subject to such a bubble regime. This test has been applied in dozens of different empirical studies. However, Hamilton (1985), Driffill and Sola (1998), and Gürkaynak (2008) noted the inherent difficulties in distinguishing financial bubbles from unobserved fundamentals.

3.5 Tipping points and financial crises.

In other models, there may be a unique equilibrium but under the right historical conditions, a small change in fundamentals can produce a huge change in observed outcomes. Such dynamics might be well-described as locally linear processes that periodically experience changes in regime. Investment dynamics constitute one possible transmission mechanism.

The right sequence of events can end up triggering a big investment decline that in turn contributes to a dramatic drop in output and an effective change in regime. Acemoglu and Scott (1997) presented a model where this happens as a result of intertemporal increasing returns, for example, if an investment that leads to a significant new discovery makes additional investments more profitable for a short time. Moore and Schaller (2002), Guo, Miao, and Morelle (2005), and Veldkamp (2005) examined different settings in which investment dynamics contribute to tipping points, often through a process of learning about current opportunities. Startz (1998) demonstrated how an accumulation of small shocks could under certain circumstances trigger a dramatic shift between alternative production technologies. Learning by market participants introduces another possible source of tipping-point or regime-shift dynamics (Hong, Stein, and Yu, 2007; Branch and Evans, 2010).

Brunnermeier and Sannikov (2014) developed an intriguing description of tipping points in the context of financial crises. They posited two types of agents, designated "experts" and "households". Experts can invest capital more productively than households, but they are constrained to borrow using only risk-free debt. In normal times, 100% of the economy's equity ends up being held by experts. But as negative shocks cause their net worth to decline, they can end up selling off capital to less productive households, lowering both output and investment. This results in a bimodal stationary distribution in which the economy spends most of its time around the steady state in which experts hold all the capital. But a sequence of negative shocks can lead the economy to become stuck in an inefficient equilibrium from which it can take a long time to recover.

A large number of researchers have used regime-switching models to study financial crises empirically. These include Hamilton's (2005) description of banking crises in the 19th century, Asea and Blomberg's study of lending cycles in the late 20th century, and an investigation of more recent financial stress by Hubrich and Tetlow (2013).

3.6 Currency crises and sovereign debt crises.

A sudden loss of confidence in a country can lead to a flight from the currency which in turn produces a shock to credit and spending that greatly exacerbates the country's problems. A sudden wave of pessimism could be self-fulfilling, giving rise to multiple equilibria that could exhibit Markov switching (Jeanne and Masson, 2000), or could be characterized by tipping point dynamics where under the right circumstances a small change in fundamentals pushes a country into crisis. Empirical investigations of currency crises using regime-switching models include Peria (2002) and Cerra and Saxena (2005).

Similar dynamics can characterize yields on sovereign debt. If investors lose confidence in a country's ability to service its debt, they will demand a higher interest rate as compensation. The higher interest costs could produce a tipping point that indeed forces a country into default or to make drastic fiscal adjustments (Greenlaw, et al., 2013). Analyses of changes in regime in this context include Davig, Leeper, and Walker (2011) and Bi (2012).

3.7 Changes in policy as the source of changes in regime.

Another source of changes in regime is a discrete shift in policy itself. One commonly studied possibility is that control of monetary policy may periodically shift between hawks and doves, the latter being characterized by either a higher inflation target or more willingness to tolerate deviations of inflation from target. Analyses using this approach include Owyang and Ramey (2004), Schorfheide (2005), Liu, Waggoner, and Zha (2011), and Bianchi (2013).

An alternative possibility is that changes in fiscal regime can be a destabilizing factor. Ruge-Murcia (1995) showed how a lack of credibility of the fiscal stabilization in 1984 contributed to the changes in inflation Israel experienced, while Ruge-Murcia (1999) documented the close connection between changes in fiscal regimes and inflation regimes for Brazil.

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Table 1

Parameter (1)	Interpretation (2)	Value from NBER classifications (3)	Value from GDP alone (4)
m_1	average growth in expansion	4.5	4.62
m_2	average growth in recession	-1.2	-0.48
σ	standard deviation of growth	3.5	3.34
p_{11}	prob. expansion continues	0.95	0.92
p_{22}	prob. recession continues	0.78	0.74

Parameter estimates based on characteristics of expansions and recessions as classified by NBER (column 3), and values that maximize the observed sample log likelihood of postwar GDP growth rates (column 4), 1947:Q2-2004:Q2. Source: Chauvet and Hamilton (2006).

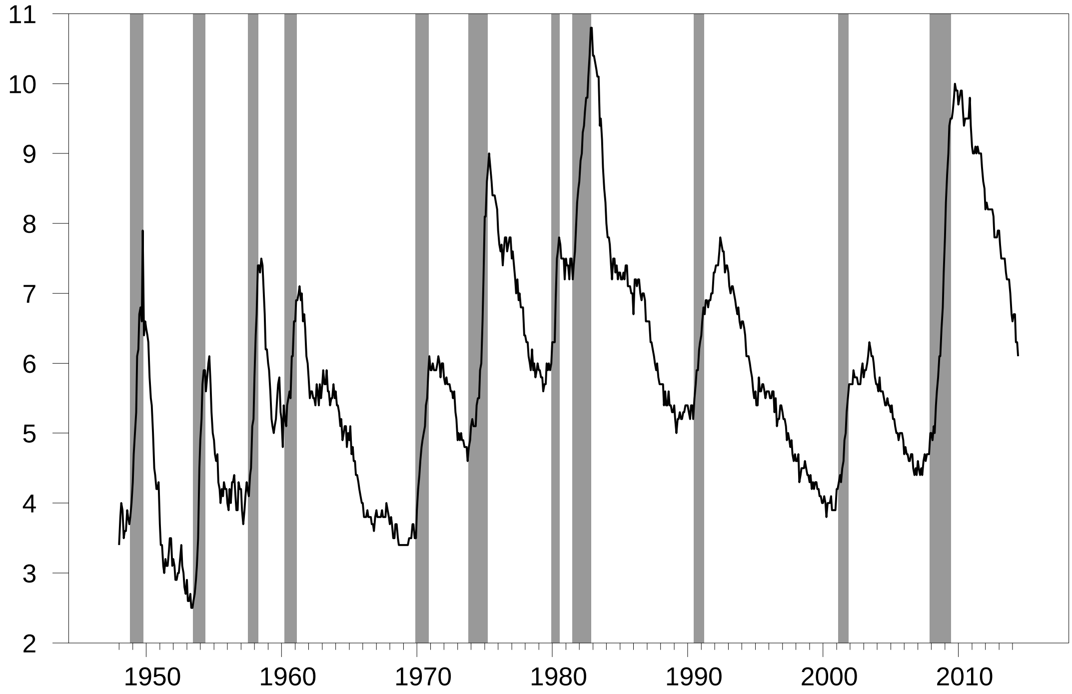

Figure 1. U.S. civilian unemployment rate, seasonally adjusted, 1948:M1-2014:M6. Shaded regions correspond to NBER recession dates.

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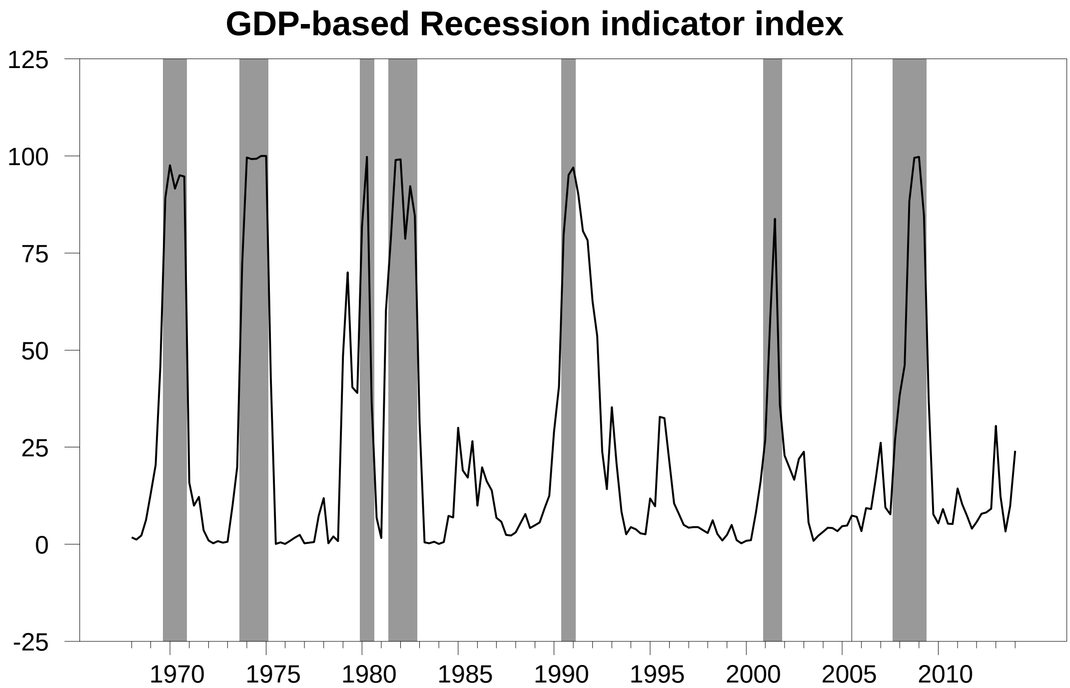

Figure 2. One-quarter-ahead smoothed probabilities $Prob(s_t = 2|\Omega_{t+1}, \hat{\vartheta}_{t+1})$, 1967:Q4-2014:Q1, as inferred using solely GDP data as reported as of date t +1. Shaded egions correspond to NBER recession dates which were not used in any way in constructing the probabilities. Prior to 2005, each point on the graph corresponds to a simulated real-time inference that was constructed from a data set as it would have been available four months after the indicated date, as reported in Chauvet and Hamilton (2006). After 2005, points on the graph correspond to actual announcements that were publicly released four months after the indicated date. Source: updated from Hamilton (2011) and www.econbrowser.com.